# MPRI 2.30: Proofs of Security Protocols

## TD: Signed Diffie-Hellman Key-Exchange

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*Questions marked with a star ($\star$) can be omitted without impacting the rest of the exercise.*

### 0.1 Signature Scheme and EUF-CMA

An signature scheme $(\mathsf{pk}, \mathsf{sk}, \mathsf{sign}, \mathsf{check})$ comprises:

- public and private key-generation functions $\mathsf{pk}(\_)$ and $\mathsf{sk}(\_)$;
- a signature function $\mathsf{sign}(\_, \_)$;
- and a signature checking function $\mathsf{check}(\_, \_, \_)$.

The public and private keys are generated from a key seed $\mathsf{n} \in \{0,1\}^\eta$ by some party $\mathsf{A}$. The public key $\mathsf{pk}(\mathsf{n})$ is shared with everybody, e.g. using a key server, while the secret key $\mathsf{sk}(\mathsf{n})$ must remain secret. The signature $\sigma = \mathsf{sign}(m, \mathsf{sk}(\mathsf{n}))$ of a message $m$ is computed using the private key $\mathsf{sk}(\mathsf{n})$, and proves that $m$ originated from $\mathsf{A}$. This signature can be verified by anyone using the public key $\mathsf{pk}(\mathsf{n})$ and the signature checking function $\mathsf{check}(\_, \_, \_)$. To this end, we must have that:

$$\forall \mathsf{n} \in \{0,1\}^\eta, \forall m. \mathsf{check}(\mathsf{sign}(m, \mathsf{sk}(\mathsf{n})), m, \mathsf{pk}(\mathsf{n})) = \mathsf{true}$$

**Remark 1.** *For the sack of conciseness, the security parameter $\eta$ has been omitted in the definitions above: Actually, all the functions above take $\eta$ as additional argument (in unary).*

**Unforgeability** A signature scheme is computationally unforgeable when no adversary can build valid signatures, even if it is provided the the public key $\mathsf{pk}(\mathsf{n})$ and has access to a signing oracle. This cryptographic assumption is the asymmetric counter-part to the unforgeability assumption MACs.

**Definition 1.** A signature scheme $(\mathsf{pk}, \mathsf{sk}, \mathsf{sign}, \mathsf{check})$ is *unforgeable against chosen-message attacks* (EUF-CMA) iff. for every PPTM $\mathcal{A}$:

$$\mathsf{Pr}_{\mathsf{n}}\left(\mathcal{A}^{\mathcal{O}_{\mathsf{sign}(\cdot,\mathsf{sk}(\mathsf{n}))}}(1^\eta, \mathsf{pk}(\eta)) = \langle m\,,\, \sigma\rangle,\ m \text{ not queried to } \mathcal{O}_{\mathsf{sign}(\cdot,\mathsf{sk}(\mathsf{n}))} \text{ and } \mathsf{check}(\sigma, m, \mathsf{sk}(\mathsf{n}))\right)$$

is negligible $\in \eta$, where $\mathsf{n}$ is drawn uniformly at random in $\{0,1\}^\eta$.

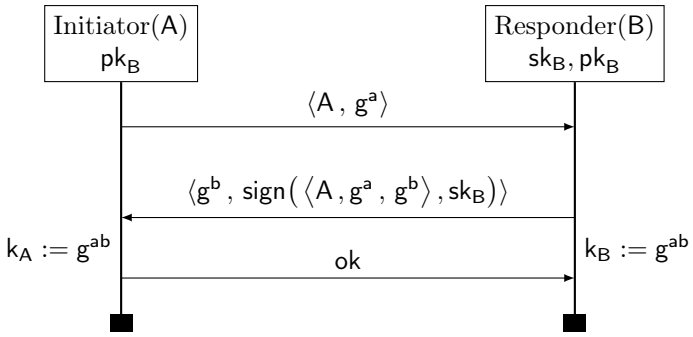
**Question 1.** *Design a rule schemata for EUF-CMA for signatures.*

# 1 Signed Diffie-Hellman

The Signed Diffie-Hellman protocol is a key-exchange protocol. This is a two party protocol, between an Initiator with identity $\mathsf{A}$ and a responder $\mathsf{B}$. The protocol aims at establishing a **shared secret** key $\mathsf{k}$ between $\mathsf{A}$ and $\mathsf{B}$. This key can then be used as a *symmetric* encryption key in future communications between $\mathsf{A}$ and $\mathsf{B}$.

Let $(\mathcal{G}, \mathsf{e}, +)$ be a finite cyclic group[1], and $\mathsf{g}$ a generator of $\mathcal{G}$. Exponentiation of an element $\mathsf{x} \in \mathcal{G}$ by $y \in \mathbb{N}$ is written $x^y := \underbrace{x + \cdots + x}_{y \text{ times}}$. The Signed-DH protocol, depicted in Figure 1, works roughly as follows:

[1] Actually a family of groups indexed by the security parameter.




**Notation:** $\mathsf{sk_B} \equiv \mathsf{sk}(\mathsf{n_B})$, $\mathsf{pk_B} \equiv \mathsf{pk}(\mathsf{n_B})$.

Figure 1: The signed Diffie-Hellman protocol Signed-DH.

- A samples uniformly at random a secret exponent a, and sends the public value $\mathsf{g^a}$ to B;
- idem for B, which samples the secret b, and sends $\mathsf{g^b}$ to A in a signed message, and computes the shared secret key $\mathsf{g^{ab}} = (\mathsf{g^a})^\mathsf{b}$;
- if the signature is valid, A computes the shared secret key $\mathsf{g^{ab}} = (\mathsf{g^b})^\mathsf{a}$ and sends ok (if the signature check fails, A sends ko).

We consider a scenario with many initiators, each running many sessions, but with a single responder B, common to all initiators. The responder B also runs many sessions.

## 1.1 Modeling

Let $\mathcal{I}$ be a finite set of identities.

**Question 2.** *Write the processes:*

- $P(A, i)$ *representing the* $i$*-th session of the initiator* $A \in \mathcal{I}$*;*
- $B(j)$ *representing the* $j$*-th session of the responder* $B$*.*

*Note that there is a single* B*, which accepts to talk to any initiator* $A \in \mathcal{I}$*.*
*We will use the channel* $A_i^0$ *and* $A_i^1$ *for* $P(A, i)$*, and* $B_j$ *for* $B(j)$*. Moreover, the random exponents sampled by* $P(A, i)$ *and* $B(j)$ *will be, respectively,* $a_{A,i}$ *and* $b_j$*.*

Let $N, M \in \mathbb{N}$. We consider the top-level process Q:

$$\nu\, \mathsf{n_B}.\ \left(!_{\mathsf{A}\in\mathcal{I}}\, !_{i\le N}\ P(\mathsf{A}, i)\right) \mid \left(!_{i\le M}\ B(j)\right)$$

**Question 3.** *For any trace* $\mathtt{tr} : A_i^1 \in \mathcal{T}_{io}$*, write a term* $\mathsf{accept}_Q@\mathtt{tr}$ *representing the acceptance check of* $P(A, i)$*. To do this, we may use* $\mathsf{in}_Q@\mathtt{tr}$*, which represents the messages inputted at the end of* $\mathtt{tr}$*.*

**Question 4.** *Give the definition of* $\mathsf{out}_Q@\mathtt{tr}$*, for any trace* $\mathtt{tr} : \mathsf{c} \in \mathcal{T}_{io}$*, where* c *is any of the channels* $A_i^0$*,* $A_i^1$ *or* $B_j$*.*

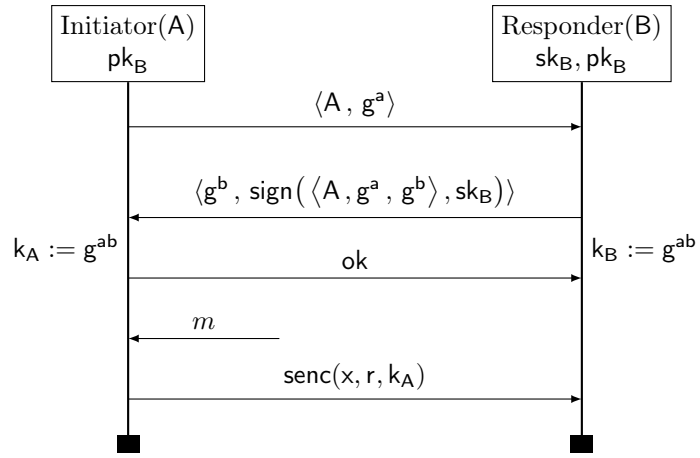
**Key-Agreement** Intuitively, the Signed-DH protocol has the key agreement property if, for any trace $\mathtt{tr} \in \mathcal{T}_{\mathsf{io}}$, for any identity A, if $P(\mathsf{A}, i)$ ends in an accepting state, then there exists a session $j$ of B such that:

- $P(\mathsf{A}, i)$ and $B(j)$ are properly interleaved;
- $P(\mathsf{A}, i)$ and $B(j)$ both derived the key $\mathsf{g}^{\mathsf{a}_{\mathbf{A},i}\,\mathsf{b}_j}$.

We are now going to translate this property into a (set of) formulas of the logic.

**Question 5.** *For any* $\mathtt{tr} : A_i^1 \in \mathcal{T}_{io}$*, write a term* $\mathsf{derived\text{-}key}_Q^A@\mathtt{tr}$ *representing the key derived by* $P(A, i)$*.*
*Similarly, write a term* $\mathsf{derived\text{-}key}_Q^B@\mathtt{tr}$ *representing the key derived by* $B(j)$*.*

**Question 6.** *Using everything above, give a set of formulas stating that the* Signed-DH *protocol has the key-agreement property for any trace* $\mathtt{tr} \in \mathcal{T}_{io}$*.*




**Notation:** $\mathsf{sk_B} \equiv \mathsf{sk}(\mathsf{n_B})$, $\mathsf{pk_B} \equiv \mathsf{pk}(\mathsf{n_B})$

Figure 2: The signed Diffie-Hellman protocol with a single message exchanged $\mathsf{Signed\text{-}DH_m}$

## 1.2 Security Proof

We are now going to prove that Signed-DH has the key-agreement property.

**Question 7.** *For any $\mathit{tr} \in \mathcal{T}_{io}$, give the set of honest signatures $\mathcal{S}$:*

$$\{m \mid \mathit{sign}(m, \mathit{sk}(n)) \in \mathit{st}(\mathit{in}_{\mathcal{O}} @ \mathit{tr})\}$$

**Question 8** ($\star$)**.** *Let $(\mathcal{G}, \mathsf{e}, +)$ be a family of cyclic groups of order $\mathsf{O}_\eta$. For any ground term $t$ and name $\mathsf{n} \in \mathcal{N}$ such that $\mathsf{n} \notin \mathit{st}(t)$, prove that the formula:*

$$[g^{\mathsf{n}} \neq t]$$

*is valid in any computational model where $\mathsf{O}_\eta$ is asymptotically large, in the sense that $1/\mathsf{O}_\eta$ is negligible.*

**Question 9.** *Prove that Signed-DH has the key-agreement property by showing that the formulas of Question 6 are valid in any computational model where:*

- *the signature scheme $(\mathit{pk}, \mathit{sk}, \mathit{sign}, \mathit{check})$ is EUF-CMA;*
- *$(\mathcal{G}, \mathsf{e}, +)$ is a family of cyclic groups of order $\mathsf{O}_\eta$ such that $1/\mathsf{O}_\eta$ is negligible.*

## 1.3 Signed DH with Message

We now go further in the modeling, and consider that Alice sends a message to Bob using the derived key and a symmetric encryption $\mathsf{senc}(m, \mathsf{r}, \mathsf{k_A})$[2]. To be as general as possible, we do not fix the content of the message Alice sends to Bob. Instead, we assume the worse, and let the adversary choose it. The protocol $\mathsf{Signed\text{-}DH_m}$ is depicted in Figure 2.

Our goal is to prove that $\mathsf{Signed\text{-}DH_m}$ is indistinguishable from an idealized version of the protocol $\mathsf{Signed\text{-}DH_m^{id}}$, where the content of the message sent has been replaced by a message of the same length, with all bits set to zero.

**Question 10.** *Write the real-world and ideal-world protocols $\mathit{Signed\text{-}DH_m}$ and $\mathit{Signed\text{-}DH_m^{id}}$.*

To do this proof, we are going to make two cryptographic assumptions. We require that:

- the symmetric encryption used satisfies the *symmetric* IND-CCA$_1^{\mathcal{G}}$ assumption;
- the group used satisfy the *Decisional Diffie-Hellman* assumption.

[2] $\mathsf{r}$ is the symmetric encryption randomness.

**Symmetric IND-CCA$_1^{\mathcal{G}}$** The symmetric IND-CCA$_1^{\mathcal{G}}$ assumption on a symmetric encryption scheme $(\mathsf{senc}(\_,\_,\_), \mathsf{sdec}(\_,\_))$ is very similar to the asymmetric one. The only differences are:

- instead of giving the public key to the adversary, it has access to an symmetric encryption oracle;
- symmetric keys are assumed to be randomly generated group elements, obtained by putting $\mathsf{g}$ to an exponent sampled uniformly at random.

We omit the precise description of the game here, and admit that the ground rule:

$$\frac{[\mathsf{len}(t_0) = \mathsf{len}(t_1)]}{\vec{u}, \mathsf{senc}(t_0, \mathsf{r}, \mathsf{g}^{\mathsf{n}}) \sim \vec{u}, \mathsf{senc}(t_1, \mathsf{r}, \mathsf{g}^{\mathsf{n}})}\ \text{IND-CCA}_1^{\mathcal{G}}$$

is sound, when:

i) $\mathsf{r} \in \mathcal{N}$ does not appear in $\vec{u}, t_0, t_1$;

ii) $\mathsf{n} \in \mathcal{N}$ appears only terms of the form $\mathsf{senc}(v, \mathsf{r}_0, \mathsf{g}^{\mathsf{n}})$ where $\mathsf{r}_0 \in \mathcal{N}$ or $\mathsf{sdec}(v, \mathsf{g}^{\mathsf{n}})$ in $\vec{u}, t_0, t_1$;

iii) for all name $\mathsf{r}_0$ such that $\mathsf{senc}(v, \mathsf{r}_0, \mathsf{g}^{\mathsf{n}})$ is a subterm of $\vec{u}, t_0, t_1$, all occurrences of $\mathsf{r}_0$ are in the subterm $\mathsf{senc}(v, \mathsf{r}_0, \mathsf{g}^{\mathsf{n}})$.

**Question 11** ($\star$)**.** *From the description and rule above, give the definition of the* IND-CCA$_1^{\mathcal{G}}$ *cryptographic assumption. Explain why item iii) is necessary for the rule soundness.*

**Decisional Diffie-Hellman** A cyclic group family $(\mathcal{G}, \mathsf{e}, +)$ satisfies the Decisional Diffie-Hellman assumption (DDH) if no adversary can distinguish values sampled from $(\mathsf{g}^a, \mathsf{g}^b, \mathsf{g}^{ab})$ from values sampled from $(\mathsf{g}^a, \mathsf{g}^b, \mathsf{g}^c)$ (where $a, b$ and $c$ are uniformly sampled at random in $\{0,1\}^\eta$) with non-negligible probability. Formally, for every PPTM $\mathcal{A}$:

$$\left| \Pr_{a,b}\left(\mathcal{A}(1^\eta, \mathsf{g}^a, \mathsf{g}^b, \mathsf{g}^{ab})\right) - \Pr_{a,b,c}\left(\mathcal{A}(1^\eta, \mathsf{g}^a, \mathsf{g}^b, \mathsf{g}^c)\right) \right|$$

must be negligible in $\eta$, when $a, b$ and $c$ are uniform samplings in $\{0,1\}^\eta$.

**Question 12** ($\star$)**.** *Give a cyclic group family such that the* DDH *assumption does not hold.*

**Question 13** ($\star$)**.** *Show that* DDH *is a stronger assumption (i.e. harder to met) than the* DLog *assumption*[3]*.*

**Question 14.** *Design a rule schemata for the* DDH *assumption. First, design the simplest rule possible capturing the* DDH *assumption.*

*Then, design a more general rule, which allows the application of the* DDH *assumption under an arbitrary context. Prove that the generalized variant is admissible from the simpler variant using standard rules of the indistinguishability logic.*

### Security of Signed-DH$_{\mathsf{m}}$

**Question 15.** *Prove that* Signed-DH$_m$ $\approx$ Signed-DH$_m^{id}$ *in any computational model where:*

- *the signature scheme* $(pk, sk, sign, check)$ *is* EUF-CMA*;*
- $(\mathcal{G}, \mathsf{e}, +)$ *is a family of cyclic groups of order* $O_\eta$ *such that* $1/O_\eta$ *is negligible.*
- *the symmetric encryption scheme* $(\mathsf{senc}(\_,\_,\_), \mathsf{sdec}(\_,\_))$ *is* IND-CCA$_1^{\mathcal{G}}$*;*
- *the group family* $(\mathcal{G}, \mathsf{e}, +)$ *satisfies the* DDH *assumption.*

[3]The discrete logarithm assumption DLog state that PPTM can compute $a$ from $\mathsf{g}^a$ with non-negligible probability, where $a$ is sampled uniformly at random.